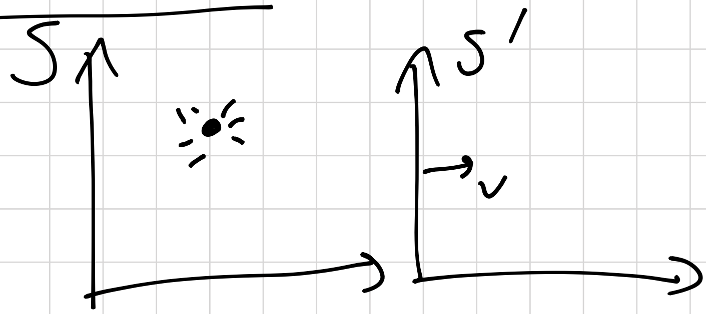
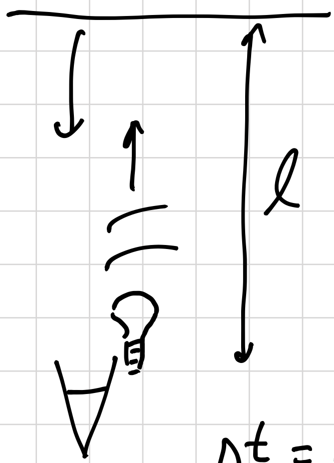
Special Relativity

Time dilation

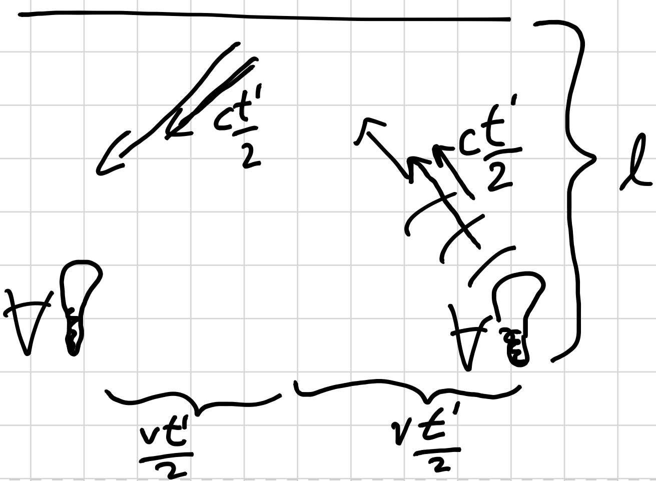


"proper time" measured between 2 events that occur in the same location



$$\Delta t_0 = t_2 - t_1 = \frac{2\ell}{c}$$

in moving frame S':



$$\ell^2 + \left(\frac{vt'}{2}\right)^2 = \left(\frac{ct'}{2}\right)^2$$

$$4\ell^2 = t'^2\left(c^2 - v^2\right)$$

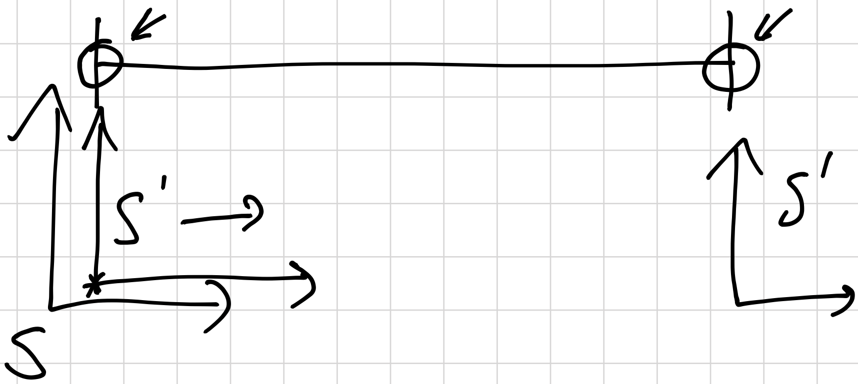
$$t'^2 = \frac{4\ell^2}{c^2 - v^2} = \frac{\cancel{4}\,\frac{\Delta t_0^2}{\cancel{4}}\,c^2}{c^2 - v^2} = \Delta t_0^2 \,\frac{1}{1 - \frac{v^2}{c^2}}$$

$$t' = \Delta t_0 \,\gamma$$

$$\gamma = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} \geq 1$$

Length contraction

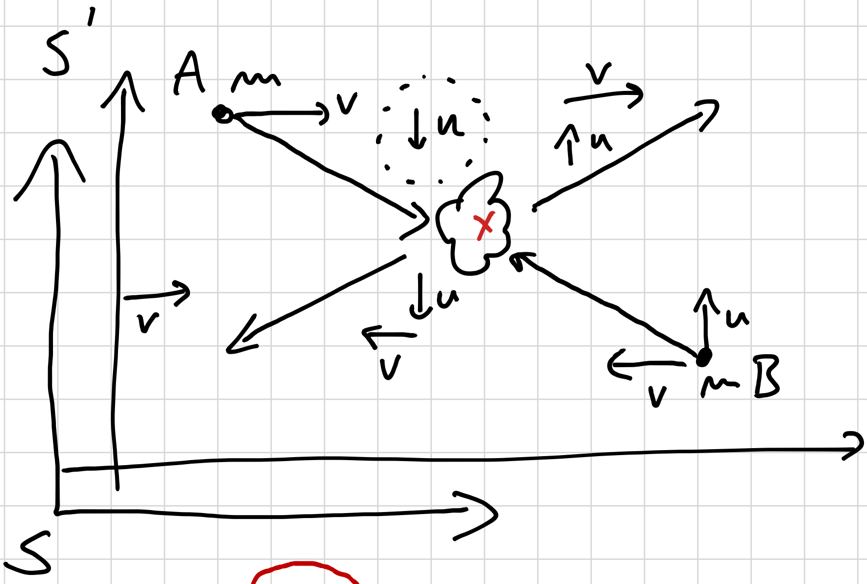
proper length: measured in frame where two end points are not moving



$$\ell = v\underbrace{\Delta t_0} = v\frac{1}{\gamma}t = \underbrace{vt}_{\ell_0}\sqrt{1 - \frac{v^2}{c^2}}$$

$$= \ell_0 \sqrt{1 - \frac{v^2}{c^2}} \longrightarrow \leq 1$$

Relativistic momentum



$$S' \left|\ \begin{array}{ll} A: & (v, -u) \to (v, +u) \\ B: & (-v, u) \to (-v, -u) \end{array}\right.$$

$$S \left|\ \begin{array}{ll} A: & (u_x^A, u_y^A) \\ B: & (0, u_y^B) \end{array}\right.$$

$$u_x' = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

$$u_y' = \frac{u_y \sqrt{1-\beta^2}}{1 - \frac{v u_x}{c^2}}$$

$$\beta = v/c$$

$$S' \left|\ \begin{array}{ll} A: & \left(\dfrac{u_x^A - v}{1 - \frac{v u_x^A}{c^2}}, \dfrac{u_y^A \sqrt{1-\beta^2}}{1 - \frac{v u_x^A}{c^2}} \right) \\ B: & \left(-v, \quad u_y^B \sqrt{1-\beta^2} \right) \end{array}\right.$$

$$\frac{u_y^A}{1 - \frac{v u_x^A}{c^2}} = -u_y^B$$

$$0 \neq m u_y^A + m u_y^B = m u_{y,\text{after}}^A + m u_{y,\text{after}}^B$$

$$\boxed{m_A = \frac{m_B}{1 - \frac{v u_x^A}{c^2}}}$$

$$m_A u_y^A + m_B u_y^B$$

$$= \frac{m_B}{1 - \frac{v u_x^A}{c^2}} u_y^A + m_B \frac{-u_y^A}{1 - \frac{v u_x^A}{c^2}} = 0$$

$$\frac{u_x^A - v}{1 - \frac{v u_x^A}{c^2}} = v$$

$$u_x^A - v = v - \frac{v^2 u_x^A}{c^2}$$

$$\frac{v^2 u_x^A}{c^2} - 2v + u_x^A = 0$$

$$v = \frac{1}{2 u_x^A} c^2 \left[2 + \sqrt{4 - 4 \frac{u_x^A}{c^2} u_x^A} \right]$$

$$= \frac{c^2}{u_x^A} \left[1 \pm \sqrt{1 - \frac{u_x^{A2}}{c^2}} \right]$$

$$\frac{1}{1-\frac{v u_x^A}{c^2}} = \frac{1}{1-\left[1 \pm \sqrt{1-\frac{u_x^{A^2}}{c^2}}\right]}$$

$$= \frac{1}{\sqrt{1-\frac{u_x^{A^2}}{c^2}}}$$

$$m_A = \frac{m_B}{\sqrt{1-\frac{u_x^{A^2}}{c^2}}}$$

as $\underline{u \to 0}$, $m_B \to m_0$

$u_x^A \to u^A$

$$\boxed{m_A = \frac{m_0}{\sqrt{1-\frac{u^{A^2}}{c^2}}} = m_0 \gamma}$$

As $u^A \to c$, $m_A \to \infty$

$$\vec{F} = \frac{d}{dt}\vec{p}$$

classical $\vec{p} = m\vec{u}$

$$\frac{d}{dt}\vec{p} = m\vec{a}$$

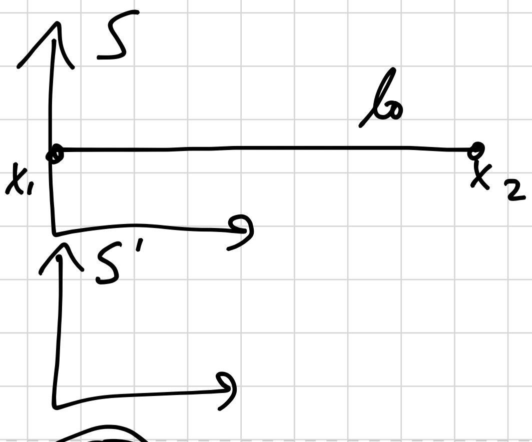
$$= \frac{d}{dt}\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\vec{u} \qquad \vec{a} := \frac{d\vec{u}}{dt}$$

$$= \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\vec{a} + \cancel{\frac{1}{2}}\frac{m_0\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}^{\,3}}\left(+\cancel{2}\,\frac{\vec{u}\cdot\frac{d\vec{u}}{dt}}{c^2}\right)$$

$$= \frac{m_0\vec{a}}{\sqrt{1-\frac{u^2}{c^2}}} + \frac{m_0\vec{u}\,(\vec{u}\cdot\vec{a})/c^2}{\sqrt{1-\frac{u^2}{c^2}}^{\,3}}$$

as $u \to c$, $|\vec{F}| \to \infty$

Length contraction



t_0'

$$l = x_2' - x_1' = (\gamma x_2 - \gamma\beta c t_2) - (\gamma x_1 - \gamma\beta c t_1)$$

$$= \gamma(x_2 - x_1) - \gamma\beta c (t_2 - t_1)$$

$$= \gamma l_0 - \gamma\beta\cancel{c}\left(\cancel{\gamma t_0'} + \gamma\beta/\cancel{c}\, x_2'\right) + \gamma\beta\cancel{c}\left(\cancel{\gamma t_0'} + \gamma\beta/\cancel{c}\, x_1'\right)$$

$$= \gamma l_0 - \gamma^2\beta^2 \underbrace{x_2' + \gamma^2\beta^2 x_1'}$$

$$= \gamma l_0 - \gamma^2\beta^2 l$$

$$l\,\underbrace{(1 + \gamma^2\beta^2)} = \gamma l_0$$

$$1 + \frac{v^2/c^2}{1 - \frac{v^2}{c^2}} = \frac{1 - v^2/c^2 + v^2/c^2}{1 - v^2/c^2} =$$

$$\frac{1}{1 - \frac{v^2}{c^2}} = \gamma^2$$

$$l\gamma^2 = \gamma l_0$$

$$\boxed{l = \frac{l_0}{\gamma}}$$